Wedge products

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The purpose of this lecture is to give a definition (rather than just a list of properties) of the determinant of an $n \times n$ matrix using a construction called the *wedge product*.

1 Wedge products

Definition 1.1. Given a vector space V, we define the *wedge product* $V \wedge V$ to be a new vector space whose elements are **linear combinations of symbols** $v \wedge w$ with $v, w \in V$. Addition and scalar multiplication in $V \wedge V$ satisfy the following properties:

(i) $v \wedge (w_1 + w_2) = v \wedge w_1 + v \wedge w_2, \quad (v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w.$

(ii) $c(v \wedge w) = (cv) \wedge w = v \wedge (cw).$

(iii) $v \wedge w = -w \wedge v$. Note that this implies that $v \wedge v = 0$ since $v \wedge v = -v \wedge v$.

Example 1.2. Let $V = \mathbb{R}^2$. Throughout this note, e_j denotes the j-th standard basis vector. We have

$$\begin{aligned}(e_1 + 2e_2) \wedge (3e_1 + 4e_2) &= (e_1 + 2e_2) \wedge 3e_1 + (e_1 + 2e_2) \wedge 4e_2 \\ &= 3e_1 \wedge e_1 + 6e_2 \wedge e_1 + 4e_1 \wedge e_2 + 8e_2 \wedge e_2 \\ &= -6e_1 \wedge e_2 + 4e_1 \wedge e_2 \\ &= -2e_1 \wedge e_2.\end{aligned}$$

To get from the second line to the third line, we have used property (iii) to conclude that $e_1 \wedge e_1 = e_2 \wedge e_2 = 0$, and $6e_2 \wedge e_1 = -6e_1 \wedge e_2$.

Example 1.3. Let $V = \mathbb{R}^3$. We have

$$\begin{aligned}&(e_1 + 2e_2 + 3e_3) \wedge (4e_1 + 5e_2 + 6e_3) \\ &= (e_1 + 2e_2 + 3e_3) \wedge 4e_1 + (e_1 + 2e_2 + 3e_3) \wedge 5e_2 + (e_1 + 2e_2 + 3e_3) \wedge 6e_3 \\ &= (4e_1 \wedge e_1 + 8e_2 \wedge e_1 + 12e_3 \wedge e_1) + (5e_1 \wedge e_2 + 10e_2 \wedge e_2 + 15e_3 \wedge e_2) \\ &\qquad + (6e_1 \wedge e_3 + 12e_2 \wedge e_3 + 18e_3 \wedge e_3) \\ &= -8e_1 \wedge e_2 - 12e_1 \wedge e_3 + 5e_1 \wedge e_2 - 15e_2 \wedge e_3 + 6e_1 \wedge e_3 + 12e_2 \wedge e_3 \\ &= -3e_1 \wedge e_2 - 6e_1 \wedge e_3 - 3e_2 \wedge e_3.\end{aligned}$$

Proposition 1.4. *If $V = \mathbb{R}^n$, then $V \wedge V$ is an $\binom{n}{2}$-dimensional vector space, with basis*

$$\{e_i \wedge e_j : i < j\}.$$

Example 1.5. Let $V = \mathbb{R}^2$, then $V \wedge V$ is one-dimensional, with basis $e_1 \wedge e_2$. We have

$$\begin{aligned}(ae_1 + be_2) \wedge (ce_1 + de_2) &= ac(e_1 \wedge e_1) + ad(e_1 \wedge e_2) + bc(e_2 \wedge e_1) + bd(e_2 \wedge e_2) \\ &= (ad - bc)(e_1 \wedge e_2).\end{aligned}$$

Here, we see the 2×2 determinant appearing already.

Definition 1.6. Give a vector space V and an integer $k \geq 1$, we define the k-th wedge product $\bigwedge^k V$ to be the vector space whose elements are linear combinations of symbols $v_1 \wedge v_2 \wedge \cdots \wedge v_k$, satisfying conditions analogous to (i)–(iii) of Definition 1.1.

Example 1.7. In $\bigwedge^3 V$, for any $u, v, w \in V$, we have

$$u \wedge (v_1 + v_2) \wedge w = u \wedge v_1 \wedge w + u \wedge v_2 \wedge w,$$

$$c(u \wedge v \wedge w) = (cu) \wedge v \wedge w,$$

$$u \wedge v \wedge w = -w \wedge v \wedge u, \quad v \wedge v \wedge w = 0.$$

Also note that $w \wedge u \wedge v = u \wedge v \wedge w$ since we can obtain $w \wedge u \wedge v$ from $u \wedge v \wedge w$ by performing 2 swaps.

Proposition 1.8. *If $V = \mathbb{R}^n$, then $\bigwedge^k V$ is an $\binom{n}{k}$-dimensional vector space, with basis*

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}.$$

In particular, for $k = n$, we have that $\bigwedge^n V$ is one-dimensional, with basis $e_1 \wedge e_2 \wedge \cdots \wedge e_n$.

2 Determinant of a linear transformation

Let V be a finite-dimensional vector space, and let $T \colon V \to V$ be a linear transformation.

Definition 2.1. We can also define the k-th wedge product of the linear transformation T itself, as the linear transformation $\bigwedge^k T \colon \bigwedge^k V \to \bigwedge^k V$ given by

$$\bigwedge^k T(v_1 \wedge v_2 \wedge \cdots \wedge v_n) = T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_n).$$

Definition 2.2. If $\dim V = n$, then $\bigwedge^n V$ is one-dimensional. Any linear transformation from a one-dimensional vector space to itself is given by scalar multiplication by some constant.

We define $\det(T)$ to be the real number such that the map $\bigwedge^k T \colon \bigwedge^n V \to \bigwedge^n V$ is multiplication by $\det(T)$.

Example 2.3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This gives the linear transformation $\mu_A \colon \mathbb{R}^2 \to \mathbb{R}^2$, and

$$\begin{aligned} \textstyle\bigwedge^2(\mu_A)(e_1 \wedge e_2) &= \mu_A(e_1) \wedge \mu_A(e_2) \\ &= (ae_1 + ce_2) \wedge (be_1 + de_2) = (ad - bc)(e_1 \wedge e_2), \end{aligned}$$

where the last step comes from Example 1.5.

Thus $\bigwedge^2(\mu_A)$ scales elements of $\mathbb{R}^2 \wedge \mathbb{R}^2$ by a factor of $ad - bc$, which agrees with the formula we have already seen for the 2×2 determinant.

Example 2.4. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\begin{aligned} &\textstyle\bigwedge^3(\mu_A)(e_1 \wedge e_2 \wedge e_3) \\ &= (a_{11}e_1 + a_{21}e_2 + a_{31}e_3) \wedge (a_{12}e_1 + a_{22}e_2 + a_{32}e_3) \wedge (a_{13}e_1 + a_{23}e_2 + a_{33}e_3) \\ &= (a_{11}a_{22}a_{33} + a_{11}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33})(e_1 \wedge e_2 \wedge e_3). \end{aligned}$$

This formula is obtained by brute force expanding the wedge product. For example, the $-a_{13}a_{22}a_{31}$ term comes from $a_{31}e_3 \wedge a_{22}e_2 \wedge a_{13}e_1 = -a_{31}a_{22}a_{13}(e_1 \wedge e_2 \wedge e_3)$. (Note the sign change.)

3 Multiplicative property of determinant

Theorem 3.1. *If $S, T \colon V \to V$ are linear transformations, then*

$$\det(S \circ T) = \det S \cdot \det T.$$

Proof. It's easy to check that $\bigwedge^n(S \circ T) = (\bigwedge^n S) \circ (\bigwedge^n T)$. Indeed,

$$\begin{aligned} \textstyle\bigwedge^n(S \circ T)(v_1 \wedge \cdots \wedge v_n) &= S \circ T(v_1) \wedge \cdots \wedge S \circ T(v_n) \\ &= S(T(v_1)) \wedge \cdots \wedge S(T(v_n)) \\ &= \textstyle\bigwedge^n S(T(v_1) \wedge \cdots \wedge T(v_n)) \\ &= \textstyle\bigwedge^n S(\bigwedge^n T(v_1 \wedge \cdots \wedge v_n)), \end{aligned}$$

for any $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n V$.

But $\bigwedge^n T$ is multiplication by $\det T$, and $\bigwedge^n S$ is multiplication by $\det S$, so their composition $\bigwedge^n(S \circ T)$ is just multiplication by $\det S \cdot \det T$. Then by definition of determinant, $\det(S \circ T) = \det S \cdot \det T$. □

4 Determinants of matrices, and properties

Definition 4.1. The determinant of an $n \times n$ matrix A is just $\det \mu_A$, and we denote it by $\det A$.

Theorem 3.1 tells us that for any two $n \times n$ matrices A and B,

$$\det(AB) = \det A \det B.$$

This is because matrix multiplication corresponds to composition of functions: $\mu_{AB} = \mu_A \circ \mu_B$.

Corollary 4.2. *If A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det A}$.

Remark 4.3. If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A, then

$$\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n = \det A \cdot (e_1 \wedge \cdots \wedge e_n). \tag{♡}$$

From this, we can immediately see that the determinant has the following behavior under *column* operations.

1.' If A' is obtained from A by swapping two columns, then $\det A' = -\det A$.

2'. If A' is obtained from A by multiplying some column by the scalar c, then $\det A' = c \det A$.

3'. If A' is obtained from A by adding a multiple of one column to another, then $\det A' = \det A$.

Lemma 4.4. *If E is the elementary matrix corresponding to*

(i) a row swap, then $\det E = -1$.

(ii) multiplying a row by c, then $\det E = c$.

(iii) adding a multiple of one row to another, then $\det E = 1$.

Proof. We will prove (iii) for the row operation $c\,\mathrm{Row}_1 + \mathrm{Row}_2 \to \mathrm{Row}\,2$. The columns of the elementary matrix E are just

$$e_1 + ce_2,\ e_2,\ e_3,\ \ldots,\ e_n.$$

Note that

$$(e_1 + ce_2) \wedge e_2 \wedge \cdots \wedge e_n = e_1 \wedge e_2 \wedge \cdots \wedge e_n + ce_2 \wedge e_2 \wedge \cdots \wedge e_n = e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

Thus, by (♡), $\det E = 1$. The remaining cases can be handled similarly. □

Since row operations are given by multiplication by an elementary matrix, the multiplicative property and Lemma 4.4 together tell us that the determinant also satisfies properties 1.–3. of Proposition 5.1.1 in Shifrin, which are what we usually use to compute.