1 True / False (5 points each)

Label each statement as true or false and give a short reason. (A single sentence or counterexample is sufficient.)

(a) A simple group of order 60 has at least two subgroups of order 5.

(b) If R and R' are integral domains, then $R \times R'$ is also an integral domain.

(c) (0) is a prime ideal of $\mathbb{Q}$.

(d) If R is a PID and I is a maximal ideal of R, then R/I is finite.

(e) Let I be a prime ideal of a ring R. Then R/I is an integral domain.

(f) There is a unique ring homomorphism from $\mathbb{Z}[x] \to R[x]$ for any ring R.

2 Examples (5 points each)

Provide an example for each of the following. (No further explanation needed.)

(a) An integral domain with a finite number of elements.

(b) A ring R and a nonzero prime ideal I of R which is not maximal.

(c) Two ideals I and J of $\mathbb{Z}$ which are coprime.

(d) A (nonzero) zero divisor in the ring $\mathbb{C}[x]/(x^2 - x)$.

(e) A proper normal subgroup of the alternating group A_4.

(f) A homomorphism $\mathbb{Z}[x] \to \mathbb{Z}[x]$ which is not an isomorphism.

3 Short answer

For each of these problems, provide a short explanation with your answer.

3.1 Maximal ideals in quotient ring

Find the number of maximal ideals in the quotient ring $\mathbb{C}[x, y]/(xy - 4y, y^2 - x^3)$.

3.2 Ideal in $\mathbb{Z}[x]$

Let $R = \mathbb{Z}[x]$ and $I = (3, 1 + x, x^2 + 5)$. Find the number of elements in R/I. Is I a prime ideal?

3.3 Ideal in $\mathbb{Z}[\sqrt{-5}]$

Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (3, 1 + \sqrt{-5})$. Find the number of elements in R/I. Is I a prime ideal?

4 Proof-based problems

For each of these problems, you should write a complete proof.

4.1 Groups of order 2026

Show that there are exactly two isomorphism classes of groups of order 2026.

4.2 The rings $\mathbb{Z}[x]/(x^2-1)$ and $\mathbb{Z}\times\mathbb{Z}$

Find all homomorphisms from $\mathbb{Z}[x]/(x^2-1) \to \mathbb{Z}\times\mathbb{Z}$. Are these two rings isomorphic?

4.3 Please remember to review the bonus problems

Let $R = \mathbb{Z}[\sqrt{-5}]$. Show that every nonzero prime ideal of R is a maximal ideal.

Additional review on ideals

1. Which of the following are ideals of $\mathbb{Z}$?
   (a) $\{0\}$
   (b) $\{1\}$
   (c) $\mathbb{Z}$
   (d) $\{0, 2, 4, 6, \dots, \}$
   (e) $\{\dots, -15, -10, -5, 0, 5, 10, 15, \dots\}$
2. Which of the following are true in the ring $\mathbb{C}[x, y]$?
   (a) $xy \in (x, y)$.
   (b) $x^2 + y + 1 \in (x, y + 1)$.
   (c) $xy + 1 \in (x, y)$.
   (d) $xy + 1 \in (x^2, y)$.
   (e) $xy + 1 \in (x + 1, y - 1)$.
3. For each of the true statements "$f \in (g, h)$" above, find $r, s \in \mathbb{C}[x, y]$ such that $f = rg + sh$.
4. Which of the following are true in the ring $\mathbb{Z}[x]$?
   (a) $3 \in (2, 4)$.
   (b) $5 \in (2x + 1, x^2 + 1)$.
   (c) $(x - 2, 5) = (2x + 1, 5)$.
   (d) $(x - 1, 3)$ is a principal ideal.
   (e) $(x^2 - x, 3x)$ is a principal ideal.
5. Find a polynomial $f \in \mathbb{Q}[x]$ such that $(f) = (x^2 + x - 2, x^3 + x^2 - x + 2)$.