

1 True / False

Label each statement as true or false and give a short reason. (A single sentence or counterexample is sufficient.)

- (a) If K/F is a Galois extension, and $F \subseteq L \subseteq K$ is an intermediate field, then $\text{Gal}(K/L)$ is a subgroup of $\text{Gal}(K/F)$.
- (b) Every Galois extension K/\mathbb{Q} of degree 6 has exactly one intermediate field of degree 3 over \mathbb{Q} .
- (c) The finite field \mathbb{F}_{27} is a degree 3 extension of \mathbb{F}_9 .
- (d) Let $f \in \mathbb{Q}[x]$ be a degree n irreducible polynomial. The splitting field of f over \mathbb{Q} has degree n over \mathbb{Q} .
- (e) Let K/F be a field extension with finite degree. Then every element of K is algebraic over F .

2 Examples

Provide an example for each of the following. (No further explanation needed.)

- (a) A Galois extension with Galois group isomorphic to $\mathbb{Z}/6\mathbb{Z}$.
- (b) A Galois extension with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
- (c) A Galois extension with Galois group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
- (d) A primitive element of the extension $\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q}$.
- (e) A primitive element of the splitting field of $x^3 - 2$ over \mathbb{Q} .

3 Short answer

For Problems 3.1 and 3.2, provide a short explanation with your answer.

3.1 Galois group of cubic

Let K be the splitting field of $x^3 - 3x - 1$ over \mathbb{Q} . What is the degree $[K : \mathbb{Q}]$?

3.2 Number of intermediate fields

According to Problem 4.1 below, the field $K = \mathbb{Q}(\cos(2\pi/41))$ is a Galois extension of \mathbb{Q} , with Galois group isomorphic to $\mathbb{Z}/20\mathbb{Z}$. How many subfields does K have?

4 Proof-based problems

For Problems 4.1 and 4.2, you should write a complete proof.

4.1 Subfield of $\mathbb{Q}(\zeta_{41})$

Let $K = \mathbb{Q}(\cos(2\pi/41))$. Show that K/\mathbb{Q} is a Galois extension, and that $\text{Gal}(K/\mathbb{Q})$ is cyclic of order 20.

4.2 Please remember to review the bonus problems

Prove that the polynomial $x^4 - 2x^2 + 9$ is reducible in $\mathbb{F}_p[x]$ for every prime p .