1 True / False (5 points each)

Label each statement as true or false. For each true statement, briefly explain why it is true (a single sentence should be enough). For each false statement, provide a counterexample.

(a) In the dihedral group D_n, the relation $r^k s r^k = s$ holds for any integer k.

True. The relation $rsr = s$ is true for any rotation r and any rotation s.

(b) If $\varphi\colon G \to G'$ is a homomorphism of groups such that $\ker(\varphi) = \{1_G\}$, then φ is injective.

True. If $\varphi(a) = \varphi(b)$, then $b^{-1}a \in \ker(\varphi)$, so $a = b$.

(c) Let G be a group, and let $a \in G$. Then the set $H = \{g \in G : gag^{-1} = a\}$ is a subgroup of G.

True. H contains 1, it contains inverses because $gag^{-1} = a$ implies $g^{-1}ag = a$, and if $g, h \in H$, then $(gh)a(gh)^{-1} = ghah^{-1}g^{-1} = a$, so H satisfies closure.

(d) The map $*\colon G \times G \to G$ given by $g * a = ag$ defines a group action of G on G.

False. It is not associative: $(gh)*a = agh \neq ahg = g*(h*a)$ unlesss g and h commute.

(e) Let G be a group, and let N be a normal subgroup of G. Then G contains a subgroup isomorphic to G/N.

False. $\mathbb{Z}$ does not contain a subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z}$, for example.

(f) Let G be a group. If $a, b, c \in G$ are elements satisfying $ab = bc$, then $a = c$.

False. For example, in the dihedral group D_3, $rs \neq sr$.

2 Examples (5 points each)

Provide an example for each of the following. (No further explanation needed.)

(a) A finite nonabelian group.

S_3

(b) A group, and an automorphism of that group which is not the map $\varphi(g) = g$.

The map automorphism of $\mathbb{Z}/3\mathbb{Z}$ which switches the two non-identity elements.

(c) A subgroup of $\mathbb{Z}/63\mathbb{Z}$ which is not equal to $\{0\}$ or the whole group $\mathbb{Z}/63\mathbb{Z}$.

$\{\overline{0}, \overline{9}, \overline{18}, \overline{27}, \overline{36}, \overline{45}, \overline{54}\}$.

(d) A coset gH of some subgroup H of S_3, such that gH has three elements and does not contain the identity element.

Using the standard presentation, the coset $yH = \{y, xy, x^2y\}$ of $H = \{1, x, x^2\}$.

(e) A permutation in S_6 with sign -1.

$(1\ 2)$.

3 Short answer

For each of these problems, provide a short explanation with your answer.

3.1 Number of homomorphisms

Find the number of homomorphisms $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Solution. Let $\varphi\colon \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ be a homomorphism, and let $\varphi(\overline{1}) = (x, y)$. Then we must have

$$4 \cdot \varphi(\overline{1}) = \varphi(4 \cdot \overline{1}) = \varphi(\overline{0}) = (\overline{0}, \overline{0}).$$

The elements $(x, y) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ which satisfy

$$4(x, y) = (\overline{0}, \overline{0})$$

are the ones for which $4x = \overline{0}$ and $4y = \overline{0}$. Both elements $x = \overline{0}, \overline{1}$ satisfy $4x = \overline{0}$, and the elements $y \in \mathbb{Z}/6\mathbb{Z}$ which satisfy $4y = \overline{0}$ are $y = \overline{0}, \overline{3}$. Thus, there are 4 possibilities for (x, y).

Note that the value of $\varphi(\overline{1}) = (x, y)$ determines the homomorphism since $\mathbb{Z}/4\mathbb{Z}$ is generated by $\overline{1}$. Furthermore, if (x, y) does satisfy $4(x, y) = (\overline{0}, \overline{0})$, then the map

$$\varphi(\overline{a}) = (ax, ay)$$

is indeed a homomorphism. So there are $\boxed{4}$ possibilities for φ.

3.2 Action of D_5 on edges of pentagon

Consider the action of D_5 on the edges of a regular pentagon $A_1A_2A_3A_4A_5$.

(a) What are the orbits of this action?

(b) What is the stabilizer of the edge A_1A_2?

Solution.

(a) The action is transitive. The only orbit is $\{A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_1\}$.

(b) No nontrivial rotations fix A_1A_2. The only reflection which fixes A_1A_2 is the reflection across OA_4 (because OA_4 also passes through the midpoint of A_1A_2), so the stabilizer is the two-element subgroup of D_5 consisting of the identity and this reflection.

4 Proof-based problems

For each of these problems, you should write a complete proof.

4.1 Index 2 subgroup is normal

Let G be a group, and suppose H is a subgroup of G such that $[G : H] = 2$. Prove that H is normal.

Solution. Let $g \in G$, and $h \in H$. By definition of normal subgroup, we want to show that $ghg^{-1} \in H$. If $g \in H$, then this is clear. Otherwise, since $[G : H] = 2$, we have the partition

$$G = H \cup gH.$$

So ghg^{-1} is either an element of H or gH.

We claim that it is not an element of gH. If it were, then $ghg^{-1} = gh'$ for some $h' \in H$, which implies $hg^{-1} = h'$, so $g = hh'^{-1}$. But this would imply $g \in H$, contradicting the assumption that $g \notin H$.

Thus, we must have $ghg^{-1} \in H$.

4.2 Please remember to review the bonus problems

Let $n > 1$ be a positive integer. Prove that n does not divide $3^n - 2^n$.

Solution. This solution is almost the same as the solution to the bonus problem on n not dividing $2^n - 1$.

Let p be the smallest prime dividing n. If $p = 2$, then clearly $p \nmid 3^n - 2^n$ since $3^n - 2^n$ is odd. Then $3^n \equiv 2^n \pmod{p}$, so

$$(3/2)^n \equiv 1 \pmod{p}.$$

Here, we use the fact that 2 has an inverse modulo p, and $3/2$ denotes the element $3 \cdot 2^{-1}$ of $(\mathbb{Z}/p\mathbb{Z})^\times$.

Let h be the order of $3/2$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Clearly, $h > 1$ since $3/2$ is not the identity. By properties of orders and Lagrange's theorem, we have $h|n$ and $h|p-1$, but since p is the smallest prime dividing n, this is impossible.