1 True / False (5 points each)

Label each statement as true or false and give a short reason. (A single sentence or counterexample is sufficient.)

(a) A simple group of order 60 has at least two subgroups of order 5.

True. The group has at least 1 Sylow 5-subgroup, which cannot be normal, so $n_5 > 1$.

(b) If R and R' are integral domains, then $R \times R'$ is also an integral domain.

False. $(1, 0)$ is a zero divisor of $\mathbb{Z} \times \mathbb{Z}$ since $(1, 0) \cdot (0, 1) = (0, 0)$.

(c) (0) is a prime ideal of $\mathbb{Q}$.

True. $\{0\}$ is closed under addition, and $0 \cdot r \in \{0\}$ for any $r \in \mathbb{Q}$.

(d) If R is a PID and I is a maximal ideal of R, then R/I is finite.

False. $\mathbb{C}[x]/(x) \cong \mathbb{C}$, which is infinite, but (x) is a maximal ideal of $\mathbb{C}[x]$ since the quotient ring is a field.

(e) Let I be a prime ideal of a ring R. Then R/I is an integral domain.

True. If $\overline{a} \cdot \overline{b} = 0$ in R/I, then $ab \in I$, so either $a \in I$ or $b \in I$, so either $\overline{a} = 0$ or $\overline{b} = 0$ in R/I

(f) There is a unique ring homomorphism from $\mathbb{Z}[x] \to R[x]$ for any ring R.

False. There is a homomorphism $x \mapsto f(x)$ for any polynomial $f(x) \in R[x]$.

2 Examples (5 points each)

Provide an example for each of the following. (No further explanation needed.)

(a) An integral domain with a finite number of elements.

$\mathbb{Z}/5\mathbb{Z}$.

(b) A ring R and a nonzero prime ideal I of R which is not maximal.

(x) is a prime ideal of $\mathbb{Z}[x]$ which is not maximal.

(c) Two ideals I and J of $\mathbb{Z}$ which are coprime.

(3) and (5).

(d) A (nonzero) zero divisor in the ring $\mathbb{C}[x]/(x^2 - x)$.

x and $x - 1$ are both zero divisors since $x(x - 1) = 0$ in this quotient ring.

(e) A proper normal subgroup of the alternating group A_4.

The subgroup $\{\text{id}, (1\ 2)\ (3\ 4), (1\ 3)\ (2\ 4), (1\ 4)\ (2\ 3)\}$.

(f) A homomorphism $\mathbb{Z}[x] \to \mathbb{Z}[x]$ which is not an isomorphism.

The homomorphism which sends $x \mapsto x^2$.

3 Short answer

For each of these problems, provide a short explanation with your answer.

3.1 Maximal ideals in quotient ring

Find the number of maximal ideals in the quotient ring $\mathbb{C}[x, y]/(xy - 4y, y^2 - x^3)$.

Solution. The maximal ideals of this ring correspond to maximal ideals of $\mathbb{C}[x, y]$ which contain $(xy - 4y, y^2 - x^3)$.

The maximal ideals of $\mathbb{C}[x, y]$ are the ideals $M_{a,b} = (x - a, y - b)$. Note that $M_{a,b}$ is the kernel of the evaluation at (a, b) map

$$\varphi\colon \mathbb{C}[x, y] \to \mathbb{C}, \quad x \mapsto a, y \mapsto b.$$

Alternatively, this can be expressed as $f \mapsto f(a, b)$.

Thus, $M_{a,b}$ contains both $xy - 4y$ and $y^2 - x^3$ if and only these two polynomials are both in the kernel of φ, so

$$ab - 4b = 0, \quad b^2 - a^3 = 0.$$

The possible values of (a, b) are thus $(0, 0)$, $(4, 8)$, and $(4, -8)$.

So the quotient ring $\mathbb{C}[x, y]/(xy - 4y, y^2 - x^3)$ has $\boxed{3}$ maximal ideals,

$$\overline{(x, y)}, \quad \overline{(x - 4, y - 8)}, \quad \overline{(x - 4, y + 8)}.$$

3.2 Ideal in $\mathbb{Z}[x]$

Let $R = \mathbb{Z}[x]$ and $I = (3, 1 + x, x^2 + 5)$. Find the number of elements in R/I. Is I a prime ideal?

Solution. We have

$$x^2 + 5 = (x + 1)(x - 1) + 3 \cdot 2,$$

so $x^2 + 5 \in (3, x + 1)$. This means that $I \subseteq (3, x + 1)$, but clearly $(3, 1 + x) \subseteq I$ as well, so $I = (3, x + 1)$.

Thus,

$$R/I \cong \mathbb{Z}[x]/(3, x + 1) \cong \mathbb{F}_3[x]/(x + 1) \simeq \mathbb{F}_3.$$

This last step can be seen by noting that $(x+1)$ is the kernel of the surjective homomorphism $x \mapsto -1$. Or, one can also use division with remainder by the monic polynomial $x + 1$ to see that the cosets of $(x + 1)$ are just

$$[(x + 1)], \quad [1 + (x + 1)], \quad [2 + (x + 1)].$$

In any ces, the quotient ring R/I has $\boxed{3}$ elements. Furthermore, it is an integral domain (it's even a field), so I is prime. (In fact, I is maximal.)

3.3 Ideal in $\mathbb{Z}[\sqrt{-5}]$

Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (3, 1 + \sqrt{-5})$. Find the number of elements in R/I. Is I a prime ideal?

Solution. We have $\mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2 + 5)$. Thus,

$$\mathbb{Z}[\sqrt{-5}]/(3, 1 + \sqrt{-5}) \cong \mathbb{Z}[x]/(3, 1 + x, x^2 + 5).$$

By the previous problem, this quotient ring is isomorphic to $\mathbb{F}_3$. So R/I again has $\boxed{3}$ elements, and I is a prime ideal because $\mathbb{F}_3$ is an integral domain.

4 Proof-based problems

For each of these problems, you should write a complete proof.

4.1 Groups of order 2026

Show that there are exactly two isomorphism classes of groups of order 2026.

Solution. Let G be a group of order $2026 = 2 \cdot 1013$. We have n_{1013} divides 2 and $n_{1013} \equiv 1 \pmod{1013}$, so $n_{1013} = 1$. Thus G contains exactly one subgroup K of order 1013, and K is normal.

Let H be any Sylow 2-subgroup. Let x be a generator of K and let y be a generator of H, so

$$x^{1013} = y^2 = 1.$$

Since $K = \{1, x, x^2, \dots, x^{1012}\}$ is normal, we have $yxy^{-1} = x^i$ for some $1 \le i \le 1012$. Since $y^2 = 1$, we must have

$$x = y^2 x y^{-2} = y(yxy^{-1})y = yx^i y^{-1} = x^{i^2},$$

where the last equality is obtained by raising both sides of $yxy^{-1} = x^i$ to the i-th power. This implies that $i^2 \equiv 1 \pmod{1013}$, so $i = 1$ or $i = 1012$.

If $i = 1$, then $yx = xy$, and G is isomorphic to $\mathbb{Z}/1013\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, via the isomorphism

$$\mathbb{Z}/1013\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to G, \quad (j, k) \mapsto x^j y^k.$$

If $i = 1013$, then $yx = x^{1012}y$, and G is isomorphic to the dihedral group D_{1013}, which has presentation $\langle r, s | r^{1013} = s^2 = 1, sr = r^{n-1}s \rangle$, via the isomorphism

$$D_{1013} \to G, \quad r^j s^k \mapsto x^j y^k.$$

Thus, there are two isomorphism classes of groups of order 2026, the class of $\mathbb{Z}/1013\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2026\mathbb{Z}$, and the class of D_{1013}.

4.2 The rings $\mathbb{Z}[x]/(x^2-1)$ and $\mathbb{Z}\times\mathbb{Z}$

Find all homomorphisms from $\mathbb{Z}[x]/(x^2-1) \to \mathbb{Z}\times\mathbb{Z}$. Are these two rings isomorphic?

Solution. By the mapping property of quotients, a homomorphism $\overline{\varphi}\colon \mathbb{Z}[x]/(x^2-1) \mapsto \mathbb{Z}\times\mathbb{Z}$ corresponds to a homomorphism $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}\times\mathbb{Z}$ such that $\varphi(x^2-1) = (0,0)$.

Since φ is a homomorphism, this is equivalent to

$$\varphi(x^2-1) = \varphi(x)^2 - (1,1) = (0,0),$$

or $\varphi(x)^2 = (1,1)$. In $\mathbb{Z}\times\mathbb{Z}$, the elements which square to $(1,1)$ are

$$(1,1), \quad (1,-1), \quad (-1,1), \quad (-1,-1).$$

Thus, there are $\boxed{4}$ homomorphisms, given by the evaluation maps

$$f \mapsto f(\pm 1, \pm 1).$$

Furthermore, since $f(1)$ and $f(-1)$ always have the same parity, none of these homomorphisms contain $(1,0)$ in their image, so none of the homomorphisms are surjective. Thus, $\mathbb{Z}[x]/(x^2-1)$ is not isomorphic to $\mathbb{Z}\times\mathbb{Z}$.

4.3 Please remember to review the bonus problems

Let $R = \mathbb{Z}[\sqrt{-5}]$. Show that every nonzero prime ideal of R is a maximal ideal.

Solution. Please see the bonus problem solution, and the note preceeding it.

Additional review on ideals

1. Which of the following are ideals of $\mathbb{Z}$?
   (a) $\{0\}$
   (b) $\{1\}$
   (c) $\mathbb{Z}$
   (d) $\{0, 2, 4, 6, \ldots,\}$
   (e) $\{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\}$
2. Which of the following are true in the ring $\mathbb{C}[x,y]$?
   (a) $xy \in (x,y)$.
   (b) $x^2+y+1 \in (x, y+1)$.
   (c) $xy+1 \in (x,y)$.
   (d) $xy+1 \in (x^2,y)$.
   (e) $xy+1 \in (x+1, y-1)$.

3. For each of the true statements "$f \in (g,h)$" above, find $r, s \in \mathbb{C}[x,y]$ such that $f = rg + sh$.

   $xy = 0 \cdot x + x \cdot y, \quad x^2 + y + 1 = x \cdot x + 1 \cdot (y+1), \quad xy + 1 = 1 \cdot (x+1) + x \cdot (y-1).$

4. Which of the following are true in the ring $\mathbb{Z}[x]$?

   (a) $3 \in (2, 4)$.

   (b) $5 \in (2x + 1, x^2 + 1)$.

   (c) $(x - 2, 5) = (2x + 1, 5)$.

   (d) $(x - 1, 3)$ is a principal ideal.

   (e) $(x^2 - x, 3x)$ is a principal ideal.

5. Find a polynomial $f \in \mathbb{Q}[x]$ such that $(f) = (x^2 + x - 2, x^3 + x^2 - x + 2)$.

   $f(x) = x + 2$.