Math 401 Problem Set 1 (due January 16, 2026)

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Problem 1. Let a, b, c be integers, not all zero. We define $\gcd(a, b, c)$ to be the largest integer which divides all of a, b, c.

Prove that if $\gcd(a, b, c) = 1$, then there exist integers x, y, z such that $ax + by + cz = 1$.

Problem 2. Let G be a group. In this problem we'll fill in some of the proofs omitted from the lecture.

(a) The group axioms guarantee that for every element a, there is an inverse b satisfying $ab = ba = 1$. Show that the inverse promised by this axiom is *unique*: that if $ab = ba = 1$ and $ab' = b'a = 1$, then $b = b'$. This justifies the notation "a^{-1}" and referring to it as "the" inverse.

(b) Prove that $(ab)^{-1} = b^{-1}a^{-1}$.

Problem 3. For each of the following sets and operations, determine whether they form a valid group. For yes, briefly explain why each axiom is true (in some cases "it's obvious" is acceptable). For no, specify which axiom fails and briefly explain why.

(a) The set of positive real numbers, under multiplication.

(b) The set of 2×2 real symmetric matrices with nonzero determinant, under multiplication.

(c) The set of 2×2 matrices with integer entries and nonzero determinant, under multiplication.

(d) The set of 2×2 matrices with integer entries and determinant equal to 1, under multiplication.

(e) The set of odd decimal digits $\{1, 3, 5, 7, 9\}$, where $a * b$ is the last decimal digit of ab.

Problem 4. How many elements of order 2 are there in each of these groups? (Note that 1013 is prime.)

(a) $\mathbb{Z}/24\mathbb{Z}$. (b) $\mathbb{Z}/25\mathbb{Z}$. (c) $(\mathbb{Z}/1013\mathbb{Z})^\times$.

Problem 5. Let $G = S_7$, $g = (1\ 2)(3\ 4)$ and $h = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. Compute the orders of gh and hg.

Problem 6. How many subgroups does $\mathbb{Z}/7\mathbb{Z} \times D_2$? (Note: D_2 is the dihedral group of order 4.)

Bonus problem (not graded). Let p be an odd prime. Prove that p divides $(p-1)! + 1$.

Bonus problem (not graded). Let $n > 1$ be a positive integer. Prove that n does not divide $2^n - 1$.