Math 401 Problem Set 3 (due January 30, 2026)

Problem 1. Let A be an $n \times n$ matrix with real coefficients, and let $\vec{b} \in \mathbb{R}^n$. Suppose there are infinitely many vectors $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$. Explain why, if the equation $A\vec{x} = \vec{b}$ has at least one solution in $\vec{x}$, then it has infinitely many solutions.

Problem 2. Determine $[G : H]$ for each of the values of G and H below.

(a) $G = \mathbb{Z}$, and $H = 10\mathbb{Z}$, the subgroup consisting of all multiples of 10.

(b) $G = \mathbb{R}^\times$, the multiplicative group of nonzero real numbers, and $H = \mathbb{R}_{>0}$, the subgroup of positive real numbers.

(c) $G = D_4$, the dihedral group with 8 elements, and $H = \langle r^2 \rangle$, the cyclic subgroup generated by r^2, where r denotes rotation by $2\pi/4$.

Problem 3. Let p be an odd prime. Show that if there exists an integer a such that $a^2 \equiv -1 \pmod{p}$, then $(\mathbb{Z}/p\mathbb{Z})^\times$ contains an element of order 4. Conclude that $p \equiv 1 \pmod{4}$.

Problem 4. Let $\varphi\colon G \to G'$ be a group homomorphism. Suppose that $|G| = 1000$ and $|G'| = 999$. Show that φ is the trivial homomorphism.

(The trivial homomorphism is the map which sends g to $1_{G'}$ for all $g \in G$.)

Problem 5. Below, as usual, the element r of D_n denotes rotation by $2\pi/n$.

(a) We saw on Problem set 2 that $\langle r \rangle$ is a normal subgroup of the dihedral group D_5. What is the quotient $D_5/\langle r \rangle$?

(b) Show that $\langle r^2 \rangle$ is a normal subgroup of D_4, the dihedral group with 8 elements. What is the quotient $D_4/\langle r^2 \rangle$?

Problem 6. (Exercise 2.12.4) Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth roots of unity. Describe the cosets of H in G explicitly. Is G/H isomorphic to G?

Bonus problem (not graded). Find the smallest integer n such that S_n contains a subgroup isomorphic to D_{1013}, the dihedral group with 2026 elements.