Math 401 Problem Set 6 (due Monday, March 2, 2026)

Problem 1.

(a) Show that $\mathbb{Z}\left[\frac{1}{2}\right] := \left\{\frac{a}{2^k} : a \in \mathbb{Z}, k \in \mathbb{Z}\right\}$ is a subring of $\mathbb{R}$, and determine its units.

(b) Show that $\mathbb{Q}[\sqrt{2}] := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.

Problem 2. Show that the characteristic of a field is either a prime number or zero.

Problem 3. Determine all ring homomorphisms

(a) from $\mathbb{Z}[x]/(x^2 + x + 1)$ to $\mathbb{C}$.

(b) from $\mathbb{Q}[\sqrt{2}]$ to $\mathbb{Q}[\sqrt{2}]$.

Problem 4. An *automorphism* of a ring R is an isomorphism from R to itself. Determine all automorphisms of $\mathbb{Z}[x]$.

Problem 5. Show that every nonzero ideal of $\mathbb{Z}[i]$ contains a nonzero integer.

Problem 6. Let $f(x) = 2x + 1$, and let $g(x) = x^2 + 1$. Show that there do not exist polynomials $q, r \in \mathbb{Z}[x]$ with $\deg r < \deg f$ such that $g(x) = f(x)q(x) + r(x)$.

Problem 7. For each of the ideals I of $\mathbb{Z}[x]$ below, determine the number of elements in the quotient ring $\mathbb{Z}[x]/I$.

(a) $(x, 5)$.

(b) $(2x, 2x^2 + 5)$.

(c) $(2x + 1, x^2 + 1)$.

Problem 8. Approximately how long did you spend on this problem set? (Round to the nearest half-hour.)