Math 401 Problem Set 7 (due Monday, March 9, 2026)

Problem 1. Let R be a ring, and let I and J be ideals of R.

(a) Show that $I \cap J$ is an ideal.

(b) Show that $I + J := \{a + b : a \in I, b \in J\}$ is an ideal.

(c) Let $R = \mathbb{Z}$, $I = (6)$, and $J = (10)$. Compute $I \cap J$ and $I + J$.

Problem 2 (Chinese Remainder Theorem)**.** Show that if m and n are coprime positive integers, then $\mathbb{Z}/mn\mathbb{Z}$ is isomorphic to the product ring $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Problem 3.

(a) Show that $\mathbb{C}[x]/(x^2 - x)$ is isomorphic to $\mathbb{C} \times \mathbb{C}$.

(b) Is $\mathbb{F}_2[x]/(x^2 - 1)$ isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$?

Problem 4. Let $R = \mathbb{Z}/12\mathbb{Z}$. How many elements does $R[x]/(2x - 1)$ have?

Problem 5. How many zero divisors are there in the ring $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$?

Problem 6. Prove that an integral domain R which contains finitely many elements is a field.

(*Hint.* For any nonzero $x \in R$, consider the elements $1, x, x^2, \ldots$ Since R is finite, some two of them must be equal. Use this to show that x has an inverse.)

Problem 7. Approximately how long did you spend on this problem set? (Round to the nearest half-hour.)