Math 401 Problem Set 9 (due Friday, April 3, 2026)

Problem 1.

(a) Express the finite group $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$ as a direct sum of cyclic groups of prime power order.

(b) Determine the number of elements of order 3 in each of $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Conclude that these two groups are not isomorphic.

Problem 2. Let V be the vector space $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ over $\mathbb{Q}$, and let $\alpha = 3 + 5\sqrt{2}$.

(a) Show that the map $\varphi \colon V \to V$ defined by $\varphi(v) = \alpha \cdot v$ is homomorphism of $\mathbb{Q}$-vector spaces. (Here, $\alpha \cdot v$ is just the usual product of α and v as real numbers.)

(b) Compute the trace and determinant of the multiplication-by-α homomorphism φ defined in part (a).

Problem 3. Suppose V is a 3-dimensional vector space over $\mathbb{F}_7$. How many elements does V have?

Problem 4. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Is the $\mathbb{Z}$-module homomorphism

$$\varphi \colon \mathbb{Z}^2 \to \mathbb{Z}^2, \quad \varphi(v) = Av$$

an isomorphism?

Problem 5. Let F be a field, and let R be any ring which is not the zero ring. Suppose $\varphi \colon F \to R$ is a ring homomorphism. Show that φ is injective.

Problem 6. Let $f \in \mathbb{Q}[x]$ be a polynomial with rational coefficients, and let $a, b \in \mathbb{Q}$. Suppose $a + b\sqrt{2}$ is a root of f. Show that $a - b\sqrt{2}$ is also a root of f.

Problem 7. An *algebraic number* is a complex number $\alpha \in \mathbb{C}$ such that there exists some monic polynomial $f \in \mathbb{Q}[x]$ with $f(\alpha) = 0$. Recall that for any $\alpha \in \mathbb{C}$, the ring $\mathbb{Q}[\alpha]$ is defined by

$$\mathbb{Q}[\alpha] \coloneqq \{a_n\alpha^n + \cdots + a_0 : a_0, \ldots, a_n \in \mathbb{Q}\}.$$

In the above definition, n can be any nonnegative integer.

Show that if α is an algebraic number, then $\mathbb{Q}[\alpha]$ is a field. (Because of this, we usually denote it by $\mathbb{Q}(\alpha)$ instead.)

Problem 8. Determine rational numbers $a, b, c \in \mathbb{Q}$ such that $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ is the multiplicative inverse of $1 + \sqrt[3]{2}$ in $\mathbb{Q}(\sqrt[3]{2})$.

Problem 9. Approximately how long did you spend on this problem set? (Round to the nearest half-hour.)